On a Ramsey-type problem of Erdős and Pach*

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Abstract

Affirmatively answering a question of Erdős and Pach from 1983, we show for some constant $C > 0$ that for any graph G on $Ck \ln k$ vertices either G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1)$.

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MSC: 05C55 (Primary) 05D10, 05D40 (Secondary).

1 Introduction

Recall that the (diagonal, two-colour) Ramsey number is defined as the smallest integer $R(k)$ for which any graph on $R(k)$ vertices is guaranteed to contain a homogeneous set of order k — that is, a set of k vertices corresponding to either a complete or independent subgraph. The search for better bounds on $R(k)$, particularly asymptotic bounds as $k \to \infty$, is a challenging topic that has long played a central role in combinatorial mathematics (see [4, 8]).

We are interested in a degree-based generalisation of $R(k)$ where, rather than seeking a clique or coclique of order k, we seek an induced subgraph of order (at least) k with high minimum degree (clique-like) or symmetrically low maximum degree (coclique-like). By gradually relaxing the degree requirement, a spectrum of Ramsey-type, or *quasi-Ramsey*, problems arise. Erdős and Pach [1] introduced these problems in 1983 and showed that there is a sharp change in behaviour at a certain point along the spectrum. More precisely, they gave good estimates for the smallest integer $R_{1/2}(k)$ such that for any graph G on $R_{1/2}(k)$ vertices either G or its complement $\overline{G}$ contains some subgraph on $\ell \geq k$ vertices with

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minimum degree at least $\frac{1}{2}(\ell-1)$. They showed that $R_{1/2}(k) = O(k \ln k)$ and $R_{1/2}(k) = \Omega(k \ln k / \ln \ln k)$, and moreover that for the corresponding problem where $\frac{1}{2}$ is replaced with some strictly larger constant c the corresponding parameter $R_c(k)$ must be at least exponential in k. (We may take $c = 1$ to recover the original Ramsey numbers.) Three of the authors recently revisited this topic together with Pach [5] to give a more refined understanding of the threshold around $\frac{1}{2}$, showing that the change from polynomial to super-polynomial growth in k occurs when one seeks a subgraph on $\ell \geq k$ vertices with minimum degree at least $\frac{1}{2}(\ell-1) + \Theta(\sqrt{(\ell-1)\ln \ell})$ (consult [5] for precise details). The problems just described relate to the so-called *variable quasi-Ramsey* numbers, whereas here we focus on the harder version, namely the *fixed quasi-Ramsey* problem where the sought subgraph is required to have *exactly* k vertices rather than at least k vertices as above.

Using a result on graph discrepancy, Erdős and Pach [1] proved that there is a constant $C > 0$ such that for any graph G on at least Ck^2 vertices either G or its complement $\overline{G}$ has an induced subgraph on (exactly) k vertices with minimum degree at least $\frac{1}{2}(k-1)$. As alluded to in the previous paragraph, they also showed (by way of an unusual random graph construction) that the previous statement does not hold with $C' k \ln k / \ln \ln k$ in place of Ck^2 for some constant $C' > 0$. They asked if it holds instead with $Ck \ln k$, as is the case for the variable quasi-Ramsey problem. Our main contribution here is to confirm this.

Theorem 1. *There exists a constant $C > 0$ such that for any graph G on $Ck \ln k$ vertices, either G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1)$.*

Although it is short, our proof of Theorem 1 has a number of different ingredients, including the use of graph discrepancy in Section 2, an application of the celebrated 'six standard deviations' result of Spencer [9] in Section 3 and a greedy algorithm in Section 4 that was inspired by similar procedures for max-cut and min-bisection. It is interesting to remark that the two discrepancy results we use are of a different nature; the one in Section 2 is an anti-concentration result while the result of Spencer is a concentration result.

2 An auxiliary result via graph discrepancy

Our first step in proving Theorem 1 will be to apply the following result. This is a bound on a variable quasi-Ramsey number which is similar to Theorem 3(a) in [5]. The idea of the proof of this auxiliary result is inspired by the sketch argument for Theorem 2 in [1], in spite of the error contained in that sketch (cf. [5]).

Theorem 2. *For any constant $\nu \geq 0$, there exists a constant $C = C(\nu) > 1$ such that for any graph G on $Ck \ln k$ vertices, G or its complement $\overline{G}$ has an induced subgraph on $\ell \geq k$ vertices with minimum degree at least $\frac{1}{2}(\ell-1) + \nu\sqrt{\ell-1}$.*

Note that the $O(k \ln k)$ quantity is tight up to an $O(\ln \ln k)$ factor by the unusual construction in [1] (cf. also Theorem 4 in [5]). The astute reader may later notice that the second-order term $\nu\sqrt{\ell-1}$ in the minimum degree guarantee of Theorem 2 can be straightforwardly improved to an $\Omega(\sqrt{(\ell-1)\ln \ln \ell})$ term. Since this does not seem to help in our results, we have omitted this improvement to minimise technicalities. On the other hand, a standard random graph construction yields the following, which certifies that the second-order term cannot be improved to a $\omega(\sqrt{(\ell-1)\ln \ln \ell})$ term.

Proposition 3. *For any $c > 0$, for large enough k there is a graph G with at least $k \ln^c k$ vertices such that the following holds. If H is any induced subgraph of G or $\overline{G}$ on $\ell \geq k$ vertices, then H has minimum degree less than $\frac{1}{2}(\ell - 1) + \sqrt{3c(\ell - 1) \ln \ln \ell}$.*

Proof. Substitute $\nu(\ell) = \sqrt{(2c \ln \ln \ell)/\ln \ell}$ into the proof of Theorem 3(b) in [5]. (We may not use Theorem 3(b) in [5] directly as stated as it needs $\nu(\ell)$ to be non-decreasing in ℓ.) $\square$

We use a result on graph discrepancy to prove Theorem 2. Given a graph $G = (V, E)$, the *discrepancy* of a set $X \subseteq V$ is defined as

$$D(X) := e(X) - \frac{1}{2}\binom{|X|}{2},$$

where $e(X)$ denotes the number of edges in the subgraph $G[X]$ induced by X. We use the following result of Erdős and Spencer [2, Ch. 7].

Lemma 4 (Theorem 7.1 of [2])**.** *Provided n is large enough and $t \in \mathbb{N}$ satisfies $\frac{1}{2}\log_2 n < t \leq n$, then any graph $G = (V, E)$ of order n satisfies*

$$\max_{S \subseteq V, |S| \leq t} |D(S)| \geq \frac{t^{3/2}}{10^3}\sqrt{\ln(5n/t)}.$$

Proof of Theorem 2. Let $G = (V, E)$ be any graph on at least $N = \lceil Ck \ln k \rceil$ vertices for a sufficiently large choice of C. We may assume that $k > \frac{1}{2}\log_2 N$ because otherwise G or $\overline{G}$ contains a clique of order k by the Erdős-Szekeres bound [3] on ordinary Ramsey numbers.

For any $X \subseteq V$ and $\nu > 0$, we define the following skew form of discrepancy:

$$D_\nu(X) := |D(X)| - \nu|X|^{3/2}.$$

We now construct a sequence $(H_0, H_1, \ldots, H_t)$ of graphs as follows. Let H_0 be G or $\overline{G}$. At step $i + 1$, we form H_{i+1} from $H_i = (V_i, E_i)$ by letting $X_i \subseteq V_i$ attain the maximum skew discrepancy D_ν and setting $V_{i+1} := V_i \setminus X_i$ and $H_{i+1} := H[V_{i+1}]$. We stop after step $t + 1$ if $|V_{t+1}| < \frac{1}{2}N$. Let $I^+ \subseteq \{1, \ldots, t\}$ be the set of indices i for which $D(X_i) > 0$. By symmetry, we may assume

$$\sum_{i \in I^+} |X_i| \geq \frac{1}{4}N. \tag{1}$$

Claim 1. *For any $i \in I^+$ and $x \in X_i$, $\deg_{H_i}(x) \geq \frac{1}{2}(|X_i| - 1) + \nu(|X_i| - 1)^{1/2}$.*

Proof. Write $|X_i| = n_i$. We are trivially done if $n_i = 1$, so assume $n_i \geq 2$. Suppose $x \in X_i$ has strictly smaller degree than claimed and set $X'_i := X_i \setminus \{x\}$. Then, since $i \in I^+$,

$$\begin{aligned} D_\nu(X'_i) &\geq e(X'_i) - \frac{1}{2}\binom{n_i - 1}{2} - \nu(n_i - 1)^{3/2} \\ &> e(X_i) - \frac{1}{2}\binom{n_i}{2} - \nu\sqrt{n_i - 1} - \nu(n_i - 1)^{3/2}. \end{aligned}$$

Note that $n_i^{3/2} > n_i^{1/2} + (n_i - 1)^{3/2}$, which by the above implies $D_\nu(X'_i) > D_\nu(X_i)$, contradicting the maximality of $D_\nu(X_i)$. $\lozenge$

Claim 1 implies that we may assume for each $i \in I^+$ that $|X_i| \le k-1$, or else we are done. This gives for any $i_1, \ldots, i_4 \in I^+$ that

$$\left(\sum_{s=1}^{4} |X_{i_s}| \right)^{3/2} \le 8(k-1)^{3/2}. \tag{2}$$

Writing $I^+ = \{i_1, \ldots, i_m\}$, we next show the following.

Claim 2. *For any $\ell \in \{1, \ldots, m-3\}$, $D(X_{i_{\ell+3}}) \le \frac{5}{6} D(X_{i_\ell})$.*

Proof. For $X \subseteq V$, let us write $\nu(X) := \nu |X|^{3/2}$ so that $D_\nu(X) = |D(X)| - \nu(X)$. For $i_1, \ldots, i_r \in I^+$, we may write $X_{i_1,\ldots,i_r} := \bigcup_{s=1}^{r} X_{i_s}$. For disjoint $X, Y \subseteq V$, we define the *relative discrepancy* between X and Y to be

$$D(X,Y) := e(X,Y) - \frac{1}{2}|X||Y|,$$

where $e(X,Y)$ denotes the number of edges between X and Y.

Now let $i, j \in I^+$ with $i < j$. Then, by the maximality of $D_\nu(X_i)$, we have $D_\nu(X_i \cup X_j) \le D_\nu(X_i)$, i.e.

$$|D(X_i) + D(X_i, X_j) + D(X_j)| - \nu(X_{i,j}) \le |D(X_i)| - \nu(X_i) = D(X_i) - \nu(X_i),$$

and hence

$$D(X_j) \le -D(X_i, X_j) + \nu(X_{i,j}). \tag{3}$$

Applying (3) (and the fact that $\nu(X_{i_{\ell+r}, i_{\ell+s}}) \le \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$ for any $r, s \in \{0,1,2,3\}$), we find that

$$D(X_{i_{\ell+1}}) + 2D(X_{i_{\ell+2}}) + 3D(X_{i_{\ell+3}}) \le -\sum_{0 \le r < s \le 3} D(X_{i_{\ell+r}}, X_{i_{\ell+s}}) + 6\nu(\textstyle\bigcup_{s=0}^{3} X_{i_{\ell+s}}). \tag{4}$$

Using $-D(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) - \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) \le D_\nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}}) \le D_\nu(X_{i_\ell})$, we obtain

$$-\sum_{s=0}^{3} D(X_{i_{\ell+s}}) - \sum_{0 \le r < s \le 3} D(X_{i_{\ell+r}}, X_{i_{\ell+s}}) \le D(X_{i_\ell}) + \nu(\textstyle\bigcup_{s=0}^{3} X_{i_{\ell+s}}),$$

which combined with (4) implies that $D(X_{i_{\ell+2}}) + 2D(X_{i_{\ell+3}}) \le 2D(X_{i_\ell}) + 7\nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$. From this, we obtain that

$$3D(X_{i_{\ell+3}}) \le 2D(X_{i_\ell}) + 8\nu(\textstyle\bigcup_{s=0}^{3} X_{i_{\ell+s}}), \tag{5}$$

where we have used the fact that $D(X_{i_{\ell+3}}) \le D(X_{i_{\ell+2}}) + \nu(\bigcup_{s=0}^{3} X_{i_{\ell+s}})$, which follows since $D_\nu(X_{i_{\ell+3}}) \le D_\nu(X_{i_{\ell+2}})$. Using the fact that the graph H_{i_s} for any $s \in \{1, \ldots, m\}$ has at least $\frac{1}{2}N \ge \frac{C}{2} k \ln k$ vertices, it follows by Lemma 4 (using our assumption on k) that there exists a subset $Y_s \subseteq V_{i_s}$ of size at most k which satisfies

$$|D(Y_s)| \ge k^{3/2} \frac{\sqrt{\ln(C \ln k)}}{10^3}.$$

However, by our choice of X_{i_s}, we have

$$\begin{aligned} D(X_{i_s}) &\geq D_\nu(X_{i_s}) \geq D_\nu(Y_s) \geq |D(Y_s)| - \nu k^{3/2} \\ &\geq k^{3/2}\left(\frac{\sqrt{\ln(C\ln k)}}{10^3} - \nu\right) \geq 2\left(8\nu\left(\bigcup_{s=0}^{3} X_{i_{\ell+s}}\right)\right), \end{aligned}$$

by (2), provided C is sufficiently large. Therefore, from (5) we find that $3D(X_{i_{\ell+3}}) \leq 2D(X_{i_\ell}) + \frac{1}{2}D(X_{i_\ell})$, proving the claim. $\lozenge$

Claim 2 now implies that $(5/6)^{(m-1)/3}D(X_{i_1}) \geq D(X_{i_m}) \geq 1$ (assuming for simplicity $m \equiv 1 \pmod 3$), which then implies

$$m - 1 \leq \frac{3\ln(D(X_{i_1}))}{\ln(6/5)} \leq \frac{6}{\ln(6/5)}\ln(k-1).$$

By (1), we deduce that at least one of the m sets X_i with $i \in I^+$ satisfies

$$|X_i| \geq \frac{N\ln(6/5)}{25\ln k}.$$

This last quantity is at least k by a choice of C sufficiently large, contradicting our assumption that $|X_i| \leq k-1$ for each $i \in I^+$. This completes the proof. $\square$

3 Subgraphs of high minimum degree via set-system discrepancy

In this section we prove, based on a well known discrepancy result of Spencer [9], that from a graph on $\ell = Ck$ vertices with minimum degree at least $\ell/2 + C'\sqrt{\ell}$ (with C' depending on C) we can select a subgraph on k vertices that has minimum degree at least $k/2$.

We start by defining the various standard notions of discrepancy that we need. Suppose $\mathcal{H} = \{A_1, \ldots, A_n\}$ where $A_i \subseteq V = [n]$. Let $\chi : V \to \{-1, 1\}$ be a colouring of V with the colours -1 and 1. For any $S \subseteq V$, we write $\chi(S) := \sum_{i \in S}\chi(i)$ and we define the *discrepancy* of $\mathcal{H}$ to be

$$\operatorname{disc}(\mathcal{H}) := \min_{\chi \in \{-1,1\}^V} \max_{S \in \mathcal{H}} \chi(S).$$

The result of Spencer [9] states that for any such $\mathcal{H}$ we have $\operatorname{disc}(\mathcal{H}) \leq 6\sqrt{\mathrm{n}}$.

For $X \subseteq V$, we define $\mathcal{H}|_X := \{A_1 \cap X, \ldots, A_n \cap X\}$. Then the *hereditary discrepancy* of $\mathcal{H}$ is defined by

$$\operatorname{herdisc}(\mathcal{H}) := \max_{\mathrm{X} \subseteq \mathrm{V}} \operatorname{disc}(\mathcal{H}|_{\mathrm{X}}).$$

The result of Spencer also immediately implies that $\operatorname{herdisc}(\mathcal{H}) \leq 6\sqrt{\mathrm{n}}$ for any $\mathcal{H}$.

Let A be the incidence matrix of $\mathcal{H}$, i.e. A is the $n \times n$ matrix given by

$$A_{ij} = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then we clearly have

$$\operatorname{disc}(\mathcal{H}) = \min_{\mathrm{x} \in \{-1,1\}^{\mathrm{V}}} \|\mathrm{Ax}\|_\infty = 2\min_{\mathrm{x} \in \{0,1\}^{\mathrm{V}}} \left\|\mathrm{A}\left(\mathrm{x} - \frac{1}{2}\mathbb{1}\right)\right\|_\infty,$$

where $\mathbb{1}$ is the all 1 vector.

Now we define the *linear discrepancy* by

$$\mathrm{lindisc}(\mathcal{H}) := \max_{c\in[0,1]^V} \min_{x\in\{0,1\}^V} \|A(x-c)\|_\infty. \tag{6}$$

Note that here we are using $\{0,1\}$-colourings again. Similarly, we define the hereditary linear discrepancy of $\mathcal{H}$ by

$$\mathrm{herlindisc}(\mathcal{H}) := \max_{X\subseteq V} \mathrm{lindisc}(\mathcal{H}|_X).$$

A result of Lovász, Spencer, and Vestergombi [7] states that $\mathrm{herlindisc}(\mathcal{H}) \le \mathrm{herdisc}(\mathcal{H})$. (Note that the factor of 2 from [7] is missing to adjust for the slightly different definition we are using.) Combining with Spencer's result, we have

$$\mathrm{lindisc}(\mathcal{H}) \le \mathrm{herlindisc}(\mathcal{H}) \le \mathrm{herdisc}(\mathcal{H}) \le 6\sqrt{n}.$$

If we set c to be the all p vector (for some $p \in [0,1]$) in (6), we obtain the following result.

Lemma 5. *Let $A_1, \ldots, A_n \subseteq V = [n]$ and $p \in [0,1]$. Then there exists $Y \subseteq V$ such that, for all $i \in [n]$,*

$$||A_i \cap Y| - p|A_i|| \le 6\sqrt{n}.$$

We use the previous lemma to prove the following result.

Lemma 6. *Suppose $G = (V,E)$ is a graph with $\ell = Pk$ vertices for some $P > 1$ and k a positive integer, and suppose*

$$\delta(G) \ge \frac{1}{2}\ell + \eta\sqrt{\ell}$$

for some $\eta > 0$. Then G has an induced subgraph H on k vertices with minimum degree

$$\delta(H) \ge \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k}.$$

Proof. Write $V = \{v_1, \ldots, v_\ell\}$, let $A_0 = V$ and for each $i \in [\ell]$ let $A_i \subseteq V$ be the neighbourhood of v_i in G. We apply Lemma 5 to the sets $A_0, \ldots, A_{\ell-1}$ with $p = (k+1+6\sqrt{\ell})/\ell$. (Note that if $p > 1$ then with a simple calculation it is easy to see we can obtain the desired graph H simply by deleting any $\ell - k$ vertices from G.) Thus there exists $Y \subseteq V$ satisfying

$$||A_i \cap Y| - p|A_i|| \le 6\sqrt{\ell}$$

for all $i \in \{0, \ldots, \ell-1\}$. Applying this for $i = 0$ and noting $A_0 \cap Y = Y$ gives

$$k+1 = p|A_0| - 6\sqrt{\ell} \le |Y| \le p|A_0| + 6\sqrt{\ell} = k+1+12\sqrt{Pk}$$

and applying it for $i \in [\ell-1]$ gives

$$\begin{aligned}|A_i \cap Y| \ge p|A_i| - 6\sqrt{\ell} \ge \frac{k}{\ell}\left(\frac{1}{2}\ell + \eta\sqrt{\ell}\right) - 6\sqrt{\ell} &= \frac{1}{2}k + \eta\frac{k}{\sqrt{\ell}} - 6\sqrt{\ell} \\ &= \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 6\sqrt{P}\right)\sqrt{k}.\end{aligned}$$

Thus Y has between $k+1$ and $k+1+12\sqrt{P}\sqrt{k}$ vertices. Let Z be an arbitrary subset of $Y \setminus \{v_\ell\}$ of size k and let $H = G[Z]$. Then since we have removed at most $12\sqrt{Pk}+1 \leq 13\sqrt{Pk}$ vertices from Y to obtain Z, we have for each $i \in [\ell-1]$ that

$$|A_i \cap Z| \geq \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k}.$$

In particular this means

$$\delta(H) \geq \frac{1}{2}k + \left(\frac{\eta}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k},$$

as desired. □

4 Proof of Theorem 1

To prove the theorem, we use as a subroutine the following algorithm, which is inspired by the greedy algorithm for max-cut or min-bisection.

Lemma 7. *Let $G = (V, E)$ be a graph of order n with $\delta(G) \geq \frac{1}{2}(n-1) + t$ for some number t. Let $\alpha \in [0,1]$ and let $a, b \in \mathbb{N}$ such that $a + b = n$. Then either there exists $A \subseteq V$ of size a such that $\delta(G[A]) \geq \frac{1}{2}a - 1 + \alpha t$, or there exists $B \subseteq V$ of size b such that $\delta(G[B]) \geq \frac{1}{2}b - 1 + (1-\alpha)t$.*

Proof. Take any $A \subseteq V$ of size a and let $B := V \setminus A$. If there exists $x \in A$ with $\deg_A(x) < \frac{1}{2}a - 1 + \alpha t$ and $y \in B$ with $\deg_B(y) < \frac{1}{2}b - 1 + (1-\alpha)t$, then move x to B and y to A, i.e. *swap* x and y. Note that when there is no such pair of vertices x, y we are done. We just need to prove that, if we keep iterating, then this procedure must stop at some point.

Consider the number of edges in $G[A]$ before and after we swap x and y. The number of edges in $G[A]$ increases by at least

$$\deg_A(y) - \deg_A(x) - 1 \geq \delta(G) - \deg_B(y) - \deg_A(x) - 1 \geq 1/2,$$

(where we subtracted 1 in case x and y are adjacent). This shows that we cannot continue to swap pairs indefinitely. □

At last we are ready to prove the main result. In fact, we prove something stronger.

Theorem 8. *There exist constants $D, D' > 0$ such that for $k \geq 2$ and any graph G on $Dk \ln k$ vertices, G or its complement $\overline{G}$ has an induced subgraph on k vertices with minimum degree at least $\frac{1}{2}(k-1) + D'\sqrt{(k-1)/\ln k}$.*

Proof. Set $\nu = 160$, $C = C(\nu)$ as defined according to Theorem 2, and $D := 4C$. Also set $D' := 1/\sqrt{D}$.

By Theorem 2, since $C \cdot 2k\ln(2k) \leq 4Ck\ln k = Dk\ln k \leq |V(G)|$, we find G or $\overline{G}$ has an induced subgraph H on $\ell \geq 2k$ vertices with $\delta(H) \geq \frac{1}{2}(\ell-1) + \nu\sqrt{\ell-1}$. If $\ell \equiv 0 \pmod{k}$ then we can and will repeatedly apply Lemma 7 to split the graph into parts whose sizes are multiples of k, eventually finding the desired subgraph. Otherwise we must take an extra application of Lemma 7 at the beginning to break off the residual vertices mod k and treat these separately, which we now do.

Let $x = \ell \bmod k$ (so $x \in \{0, \ldots, k-1\}$). We can now apply Lemma 7 to H with $a = k + x$, $b = \ell - k - x$, $t = \nu\sqrt{\ell-1}$ and $\alpha = 1/2$. Suppose this gives us a subset $A \subseteq V(H)$ of size a such that

$$\delta(H[A]) \geq \frac{1}{2}a - 1 + \frac{1}{2}\nu\sqrt{\ell-1} \geq \frac{1}{2}a + \frac{1}{4}\nu\sqrt{\ell} \geq \frac{1}{2}a + \frac{1}{4}\nu\sqrt{a}.$$

Then $k \le a < 2k$ and, so applying Lemma 6 (with $P = a/k \in [1,2]$ and $\eta = \nu/4 = 40$) yields a subset $A' \subseteq A$ of size k such that

$$\delta(H[A']) \ge \frac{1}{2}k + \left(\frac{40}{\sqrt{P}} - 19\sqrt{P}\right)\sqrt{k} \ge \frac{1}{2}k + \left(\frac{40}{\sqrt{2}} - 19\sqrt{2}\right)\sqrt{k} \ge \frac{1}{2}k + \sqrt{2k},$$

which is more than required. In case Lemma 7 does not produce such a set A, it gives instead a subset B of size $b = \ell - k - x \equiv 0 \pmod{k}$ such that $\delta(H[B]) \ge \frac{1}{2}(b-1) + \frac{1}{2}\nu\sqrt{\ell-1} - \frac{1}{2}$. We iteratively apply Lemma 7 to $H[B]$ in a binary search to find a desired induced subgraph as follows.

Set $G_0 = H[B]$. Let $\ell_0 := |V(G_0)| = b$ (so that $k \le \ell_0 \le Dk\ln 2k$ and $\ell_0 \equiv 0 \pmod{k}$) and set $t_0 := \frac{1}{2}\nu\sqrt{\ell-1} - \frac{1}{2} \ge \frac{1}{2}\nu\sqrt{\ell_0-1} - \frac{1}{2}$ (so that $\delta(G_0) \ge \frac{1}{2}(\ell_0 - 1) + t_0$). Suppose that G_i is given, where G_i has ℓ_i vertices with $\ell_i \equiv 0 \pmod{k}$ and $\delta(G_i) \ge \frac{1}{2}(\ell_i - 1) + t_i$ for some number t_i. Set $a_i = \lfloor \ell_i/2k \rfloor k$ and $b_i = \lceil \ell_i/2k \rceil k$ so that $a_i + b_i = \ell_i$ and $a_i \equiv b_i \equiv 0 \pmod{k}$. Apply Lemma 7 with $G = G_i$, $a = a_i$, $b = b_i$, $t = t_i$, and $\alpha = \frac{1}{2}$. Then we either obtain a set of vertices A_i of size a_i such that $\delta(G_i[A_i]) \ge \frac{1}{2}a_i - 1 + \frac{1}{2}t_i$, in which case we set $G_{i+1} := G_i[A_i] = H[A_i]$, or we obtain a set of vertices B_i of size b_i such that $\delta(G_i[B_i]) \ge \frac{1}{2}b_i - 1 + \frac{1}{2}t_i$, in which case we set $G_{i+1} := G_i[B_i] = H[B_i]$. Now set $\ell_{i+1} = |V(G_{i+1})|$ and note that $\ell_{i+1} \equiv 0 \pmod{k}$ and $\delta(G_{i+1}) \ge \frac{1}{2}(\ell_{i+1} - 1) + t_{i+1}$, where $t_{i+1} = \frac{1}{2}(t_i - 1)$. Note also that $\ell_{i+1}/k \le \lceil \ell_i/2k \rceil$.

In this way we obtain subgraphs $G_0, G_1, \ldots$ of $G_0 = H[B]$ and we see from the recursion for ℓ_i above that if $\ell_i > k$ then $\ell_{i+1} < \ell_i$. Thus there exists some j such that $\ell_j = k$ (since $\ell_i \equiv 0 \pmod{k}$ for all i) and an easy computation shows we can assume that $j \le \log_2(\ell_0/k) + 1$. The recursion for t_i implies that $t_i \ge t_0 2^{-i} - 1$ so that

$$t_j \ge \frac{t_0 k}{2\ell_0} - 1 \ge \frac{\nu(\sqrt{\ell_0 - 1} - 1)k}{4\ell_0} \ge \frac{k}{\sqrt{\ell_0}} \ge \frac{\sqrt{k}}{\sqrt{D\ln k}} = D'\sqrt{\frac{k}{\ln k}}$$

(where we used that $t_0 \ge \frac{1}{2}\nu\sqrt{\ell_0 - 1} - \frac{1}{2}$, that $\ell_0 \ge k \ge 2$ with $\nu = 160$, and that $\ell_0 \le Dk\ln k$). Thus G_j has k vertices and minimum degree at least $\frac{1}{2}(k-1) + D'\sqrt{(k-1)/\ln k}$ and is an induced subgraph of $H[B]$ and hence of G or $\overline{G}$. $\square$

5 Concluding remarks

It is tempting to try using the greedy subroutine (Lemma 7) in a binary search on the output of Theorem 3(a) of [5], but since we cannot control the order of this output graph, the search might require $O(\ln k)$ steps, which would destroy the minimum degree bounds.

Determination of the second-order term in the minimum degree threshold for polynomial to super-polynomial growth of the fixed quasi-Ramsey numbers is an open problem. (The corresponding term for the variable quasi-Ramsey numbers was determined in [5].) We define the fixed quasi-Ramsey number as the least integer $R_c^*(k)$ such that for any graph G on $R_c^*(k)$ vertices either G or its complement $\overline{G}$ contains some subgraph on (exactly) k vertices with minimum degree at least $c(k-1)$. By Theorem 8 if $c - \frac{1}{2} = O(\sqrt{1/(k-1)\ln k})$ then $R_c^*(k)$ is polynomial in k, and by Proposition 3 if $c - \frac{1}{2} = \omega(\sqrt{\ln\ln k/(k-1)})$ then $R_c^*(k)$ is superpolynomial in k. Hence the choice of $c - \frac{1}{2}$ for which we find a transition between polynomial and super-polynomial growth in k of $R_c^*(k)$ is determined to within a $O(\sqrt{\ln k \ln\ln k})$ factor of $\sqrt{1/(k-1)}$. What is it precisely?

Last, we remark that, in the above notation, our main result is that $R^*_{1/2}(k) = O(k \ln k)$, while Erdős and Pach showed that $R^*_{1/2}(k) = \Omega(k \ln k / \ln \ln k)$. They also asked if $R^*_{1/2}(k) = \Omega(k \ln k)$. This question remains open.

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